AIAA JOURNAL
Vol. 47, No. 3, March 2009

Problem Formulations and Treatment of Uncertainties in Aerodynamic Design

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DOI: 10.2514/1.37955

Recently, optimization has become an integral part of the aerodynamic design process chain. Besides standard optimization routines, which require some multitude of the computational effort necessary for the simulation only, fast optimization methods based on one-shot ideas are also available, which are only 4 to 10 times as costly as one forward flow simulation computation. However, the full potential of mathematical optimization can only be exploited, if optimal designs can be computed, which are robust with respect to small (or even large) perturbations of the optimization set-point conditions. That means the optimal designs computed should still be good designs, even if the input parameters for the optimization problem formulation are changed by a nonnegligible amount. Thus even more experimental or numerical effort can be saved. In this paper, we aim at an improvement of existing simulation and optimization technology, developed in the German collaborative effort MEGADESIGN, so that numerical uncertainties are identified, quantized, and included in the overall optimization procedure, thus making robust design in this sense possible. These investigations are part of the current German research program MUNA.

Nomenclature

A	=	approximation matrix
B	=	reduced Hessian
C	=	covariance matrix
c	=	equality constraint
f	=	objective function
g	=	reduced gradient
h	=	inequality constraint
$\mathcal{L}$	=	Lagrangian function
$\mathcal{P}$	=	probability measure
p	=	design variables
P_0	=	probability
S	=	pertubation set
s	=	random variable
s^0	=	mean value of s
st	=	subject to
y	=	state vector
γ	=	reduced gradient
λ	=	Lagrange multiplier
μ	=	Lagrange multiplier
ν	=	mean value
σ	=	variance
φ	=	Lebesgue density
ω_i	=	quadrature weights
$\mathbf{1}$	=	indicator function

I. Introduction

UNCERTAINTIES pose problems for the reliability of numerical computations and their results in all technical contexts one can think of. They have the potential to render worthless even highly sophisticated numerical approaches, because their conclusions do not realize in practice due to unavoidable variations in problem data. The proper treatment of these uncertainties within a numerical context is a very important challenge. This paper is devoted to the enhancement of highly efficient optimal design techniques developed in the framework of MEGADESIGN‡ by a robustness component, which tries to make the optimal design generated a still good design, if the setting of a specific design point is varied. The investigations presented here are part of the research effort MUNA§, which has recently started.

Zang et al. [1] give an overview of existing methods and algorithms in uncertainty quantification and in robust design of engineering systems. Because the resulting robust optimization tasks become much more complex than the usual single set-point case, most of the techniques developed so far pertain to problems with a low degree of nonlinearity [2,3]. Putko et al. [4] present results of the approximate statistical moment method for uncertainty propagation and robust optimization for a simple quasi-one-dimensional Euler CFD (computational fluid dynamics) problem. Numerical results for a two-dimensional airfoil shape optimization problem are obtained using a numerical evaluation of the expectation integral and a second-order statistical moment method are compared in [5]. Here, we try to give some insight into the sources of uncertainties and their range and compare approaches for their proper treatment on a two-dimensional problem close to a real configuration.

II. The Nature of Uncertainties in Aerodynamic Design

For most of what follows it will be enough to consider a rather abstract but generic form of an aerodynamic shape optimization problem:

$$\min_{y,p} f(y, p) \tag{1}$$

$$\text{st} \;\; c(y, p) = 0 \tag{2}$$

$$h(y, p) \geq 0 \tag{3}$$

Received 8 April 2008; revision received 19 November 2008; accepted for publication 19 November 2008.

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‡MEGADESIGN Aerodynamic simulation and optimization in aircraft design, German national project funded by the German Federal Ministry of Economics and Labour (BMWA)

§MUNA Management and Minimization of Uncertainties in Numerical Aerodynamics, German national project funded by the German Federal Ministry of Economics and Labour (BMWA)

We think of Eq. (2) as the discretized outer flow equation around (e.g., an airfoil described by geometry parameter $p \in \mathbb{R}^{n_p}$). The vector y is the state vector (velocities, pressure, etc.) of the flow model Eq. (2), and we assume that Eq. (2) can be solved uniquely for y for all reasonable geometries p. The objective in Eq. (1) f: $(y, p) \mapsto f(y, p) \in \mathbb{R}$ typically is the drag to be minimized. The restriction in Eq. (3) typically denotes lift or pitching moment requirements. To make the discussion here simpler, we assume a scalar valued restriction, that is, $h(y, p) \in \mathbb{R}$. The generalization of the discussions following to more than one restriction is straight forward. In contrast to previous papers on robust aerodynamic optimization, we treat the angle of attack as a fixed parameter that is not adjusted to reach the required lift (cf. [5–7]). Therefore, we present in the following robust optimization strategies to solve the lift-constrained drag minimization problem with respect to uncertain parameters.

Uncertainties arise in all aspects of aerodynamic design. However, we want to limit the discussion here to uncertainties that cannot be avoided at all before constructing a aircraft. We distinguish two types of uncertainties: uncertainties with respect to the flight conditions and geometry uncertainties. The main characteristics of the macroscopic flight conditions are the angle of incidence and the velocity (Mach number) of the plane. One generally knows the rough values for these characteristics, but nevertheless, there will be unavoidable deviations from the nominal flight condition. In the numerical discussion following, we focus on the Mach number as an uncertain parameter within limits. We assume (mainly due to lack of statistical data) a truncated normal distribution of the perturbations with the nominal Mach number as an expected value. The resulting robust problem formulations discussed in the following require more computational effort but can be reformulated as deterministic problems similar to Eqs. (1–3).

Geometry uncertainties, on the other hand, require changing the optimal design problem dramatically. With geometry uncertainties, we mean the case that the real geometry deviates from the planned geometry characterized typically by splines parameterized by p. The parameters p span a space of possible geometries of dimension n_p. The sources for deviations from the planned geometry may lie in manufacturing, usage, and wearing of the aircraft or weather conditions (e.g., ice crusts). The only sure information about these deviations is that they will not lie within the geometry space spanned by the spline parameters p. Here, one rather has to work in the shape space, which is in general a function space that requires at least the usage of a free node parameterization. The ultimate goal of these investigations is the robust design under moderate shape fluctuations from a function space still to be determined. We leave these discussions to subsequent publications.

III. Robust Formulations of Aerodynamic Design Problems

The general deterministic problem formulation in Eqs. (1–3) is influenced by stochastic perturbations. We assume that there are uncertain disturbances $s \in S \subset \mathbb{R}^n$ involved in the form of random variables associated with a probability measure $\mathcal{P}$ with Lebesgue density φ: $S \to \mathbb{R}_0^+$ such that the expected value of s can be written as

$$E(s) = \int_S s \, \mathrm{d}\mathcal{P}(s) = \int_S s\varphi(s) \, \mathrm{d}s$$

and the expected value of any function g: $S \to \mathbb{R}$ is written as

$$E(g) = \int_S g(s) \, \mathrm{d}\mathcal{P}(s) = \int_S g(s)\varphi(s) \, \mathrm{d}s$$

The dependence can arise in all aspects, and a naive stochastic variant might be rewritten as

$$\min_{y,p} f(y, p, s) \tag{4}$$

$$\text{st} \;\; c(y, p, s) = 0 \tag{5}$$

$$h(y, p, s) \geq 0 \tag{6}$$

This formulation still treats the uncertain parameter as an additional fixed parameter. The optimal solution should be stable with respect to stochastic variations in s. The literature can be classified in the following ideal classes: min–max formulation, semi-infinite formulation, and chance constraints.

A. Min–Max Formulations

The min–max formulation aims at the worst-case scenario:

$$\min_{y_s, p} \max_{s \in S} f(y_s, p, s) \tag{7}$$

$$\text{st} \;\; c(y_s, p, s) = 0, \qquad \forall \; s \in S \tag{8}$$

$$h(y_s, p, s) \geq 0, \qquad \forall \; s \in S \tag{9}$$

Because the state vector y depends on the uncertain parameter s, there is a different y_s for each s. The min–max formulation is obviously independent of the stochastic measure $\mathcal{P}$ and, thus, needs only the perturbation set S as input. If the probability density function of the uncertain parameter is not available, this approach could potentially be an attractive strategy. Otherwise, this formulation will ignore problem-specific information, if it is at hand and will lead to overly conservative designs. We do not treat this formulation furthermore in this paper.

B. Semi-Infinite Formulations

The semi-infinite reformulation aims at optimizing the average objective function but maintaining the feasibility with respect to the constraints everywhere. Thus, it aims at an average optimal and always feasible robust solution. The ideal formulation is of the form

$$\min_{y_s, p} \int_S f(y_s, p, s) \, \mathrm{d}\mathcal{P}(s) \tag{10}$$

$$\text{st} \;\; c(y_s, p, s) = 0, \qquad \forall \; s \in S \tag{11}$$

$$h(y_s, p, s) \geq 0, \qquad \forall \; s \in S \tag{12}$$

Because the state vector y depends on the uncertain parameter s as in the min–max formulation, there is a different y_s for each s. This definition of robustness can also be found in [5,8]. Semi-infinite optimization problems have been treated directly so far only for rather small and weakly nonlinear problems, such as [9]. For the numerical treatment of complicated design tasks, one has to approximate the integral in the objective Eq. (10). If one assumes that the random variable confers to a multivariate truncated normal distribution, that is$s \sim \frac{1}{\text{const}} N(\nu, C) \cdot \mathbf{1}_S$, with expected value vector ν, covariance C, and indicator function

$$\mathbf{1}_S(x) = \begin{cases} 1, & \text{if } x \in S \\ 0, & \text{if } x \notin S \end{cases}$$

the integral in Eq. (10) can be efficiently evaluated by a Gaussian quadrature for small stochastic dimensions, where the quadrature points $\{s_i\}_{i=1}^N$ are the roots of a polynomial belonging to a class of orthogonal polynomials. Because of the exponential growth of the effort with increasing dimension, the full tensor product Gaussian quadrature rule should be replaced in the higher dimensional case by Smolyak-type algorithms, which use a recursive contribution of lower-order tensor products to estimate the integral [10]. In the case of a lift constraint in Eq. (12) to be satisfied overall within a set of

Mach numbers, we can take advantage of the fact that the lift is monotonically increasing with the Mach number. Consequently, it is enough to keep a lift constraint for the smallest Mach number under consideration. Therefore, we can reformulate the problem in Eqs. (10–12) in an approximate fashion in the form of a multiple set-point problem for the set points $\{s_i\}_{i=1}^N$:

$$\min_{y_i,p} \sum_{i=1}^{N} f(y_i, p, s_i)\omega_i \tag{13}$$

$$\text{st } c(y_i, p, s_i) = 0, \qquad \forall\ i \in \{1, \ldots, N\} \tag{14}$$

$$h(y_i, p, s_{\min}) \geq 0 \tag{15}$$

where ω_i denote the quadrature weights. We will investigate this formulation later on. As an example, we look at the Mach number as being the uncertainty s, which is scalar valued [i.e. $s \sim \frac{1}{\text{const}} N(\nu, \sigma^2) \cdot \mathbf{1}_S$] and we choose just four Gaussian points. Figure 1 shows a particular choice for the density function of the Mach number with expected value $s^0 = 0.73$ and the computed Gaussian points. So we achieve feasibility over the whole range of Mach numbers independent from the discretization in the probability space and at a given angle of attack. The robust formulations in [5,6] differ in this point from Eqs. (13–15), because the constraints on the lift are eliminated by choosing an appropriate value for the angle of attack for each realization s_i of the Mach number.

C. Chance Constraint Formulations

Chance constraints leave some flexibility with respect to the inequality restrictions (cf. [11]). The inequality restrictions are only required to hold with a certain probability $\mathcal{P}_0$:

$$\min_{y_s,p} \int_S f(y_s, p, s)\, \mathrm{d}\mathcal{P}(s) \tag{16}$$

$$\text{st } c(y_s, p, s) = 0, \qquad \forall\ s \in S \tag{17}$$

$$\mathcal{P}(\{s | h(y_s, p, s) \geq 0\}) \geq \mathcal{P}_0 \tag{18}$$

So far, chance constraints are used mainly for weakly nonlinear optimization problems [2,12]. In the context of structural optimization (which is typically a bilinear problem), this formulation is also called reliability-based design optimization. For more complex problems, we need again some simplification. In [4] this is performed by applying a Taylor series expansion about a nominal set point $s^0 := \nu$, which is, at the same time, the expected value of the random variable s. Suppressing further arguments (y, p) for the moment, the Taylor approximation of the second order of f in Eq. (16) gives

$$\hat{f}(s) := f(s^0) + \frac{\partial f(s^0)}{\partial s}(s - s^0) + \frac{1}{2}(s - s^0)^\top \frac{\partial^2 f(s^0)}{\partial s^2}(s - s^0)$$

Integrating this, we observe

$$\int_S \hat{f}(s)\, \mathrm{d}s = f(s^0) + \frac{1}{2}\sum_{i=1}^{k} \frac{\partial^2 f(s^0)}{\partial s_i^2} \mathrm{Var}(s_i)$$

where $\mathrm{Var}(s_i)$ is the variance of the ith component of s. Obviously, a first-order Taylor series approximation would not give any influence of the stochastic information, which is the reason why we use an approximation of second order for the objective. To deal with the probabilistic chance constraint Eq. (18), we have to approximate its probability distribution by something simple (e.g., again a truncated normal distribution). Therefore, we use a first-order Taylor approximation there, because we know that this is again a truncated normal distribution, unlike the second-order approximation (cf. [13]):

$$\hat{h}(s) := h(s^0) + \frac{\partial h(s^0)}{\partial s}(s - s^0) \sim \frac{1}{\text{const}} N(h(s^0), \sigma_h^2) \cdot \mathbf{1}_{S_h}$$

where we assume for simplicity that h is scalar valued.

Now we can put the Taylor approximations together and achieve a deterministic single set-point optimization problem. Because the flow model in Eq. (17) depends also on the uncertainties s, we should be aware that the derivatives with respect to s mean total derivatives. We express this by reducing the problem in writing $y = y(p, s)$ via Eq. (17):

$$\min_p f(y(p, s^0), s^0) + \frac{1}{2}\sum_{i=1}^{k} \frac{\partial^2 f(y(p, s^0), s^0)}{\partial s_i^2} \mathrm{Var}(s_i) \tag{19}$$

$$\text{st } \mathcal{P}(\{s | \hat{h}(y(p, s), s) \geq 0\}) \geq \mathcal{P}_0 \tag{20}$$

For the computation of the total derivatives we can introduce a sensitivity equation as in [14].

As an example, again we look at the case that s is scalar valued, that is, $s \sim \frac{1}{\text{const}_s} N(\nu, \sigma^2) \cdot \mathbf{1}_{[l,u]}$, where $\text{const}_s = \int_l^u \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right) \mathrm{d}x$ is the scaling factor to normalize the density function. Hence, we obtain the distribution of the probabilistic constraint:

$$\hat{h}(s) \sim \frac{1}{\text{const}_{\hat{h}}} N\left(h(s^0), \left(\frac{\partial h(s^0)}{\partial s}\right)^2 \sigma^2\right) \cdot \mathbf{1}_{\left[\frac{\partial h(s^0)}{\partial s} l + h(s^0), \frac{\partial h(s^0)}{\partial s} u + h(s^0)\right]}$$

where $\text{const}_{\hat{h}} = \frac{1}{\sqrt{2\pi(\frac{\partial h(s^0)}{\partial s})^2\sigma^2}} \int_{\frac{\partial h(s^0)}{\partial s} l + h(s^0)}^{\frac{\partial h(s^0)}{\partial s} u + h(s^0)} \exp\left(-\frac{(x - h(s^0))^2}{2(\frac{\partial h(s^0)}{\partial s})^2\sigma^2}\right) \mathrm{d}x$.

Finally, the following equivalent representations of the chance constraint:

$$\mathcal{P}(\{s | \hat{h}(y(p, s), s) \geq 0\}) \geq \mathcal{P}_0 \Leftrightarrow \mathcal{P}(\{s | \hat{h}(y(p, s), s) \leq 0\}) \leq 1 - \mathcal{P}_0$$

lead to the deterministic optimization problem

$$\min_p f(y(p, s^0), s^0) + \frac{1}{2}\sum_{i=1}^{k} \frac{\partial^2 f(y(p, s^0), s^0)}{\partial s_i^2} \mathrm{Var}(s_i) \tag{21}$$

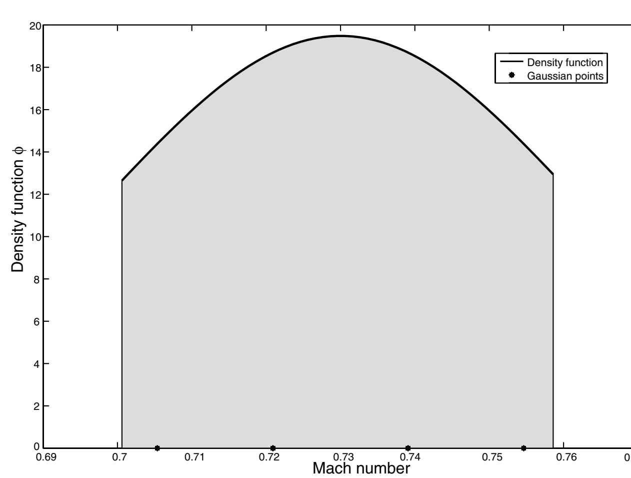

Fig. 1 Gaussian points and the density function.

$$\text{st} \quad \frac{1}{\mathrm{const}_{\hat{h}}\sqrt{2\pi(\frac{\partial h(s^0)}{\partial s})^2\sigma^2}} \int_{\frac{\partial h(s^0)}{\partial s}l+h(s^0)}^{0} \exp\left(-\frac{(x-h(s^0))^2}{2(\frac{\partial h(s^0)}{\partial s})^2\sigma^2}\right)\mathrm{d}x$$
$$\leq 1-\mathcal{P}_0 \tag{22}$$

The propagation of the input data uncertainties is estimated by the combination of a first-order second-moment method and a second-order second-moment method, presented for example in [4].

Because there is no closed-form solution for the integral, the chance constraint is evaluated by a numerical quadrature formula. In the following, we will compare the chance constraint formulation evaluating the objective Eq. (16) by a numerical quadrature formula with the formulation in Eq. (21) where the objective omits a Taylor expansion.

IV. One-Shot Aerodynamic Shape Optimization and its Coupling to Robust Design

Novel one-shot aerodynamic shape optimization in the form of Eqs. (1–3) has been introduced in [15,16]. This has the potential for fast convergence in only a small multiple of central processing unit time compared with one flow simulation. These methods are based on approximate reduced sequential quadratic programming (SQP) iterations in order to generate a stationary point satisfying the first-order Karush–Kuhn–Tucker optimality conditions.

In this context, a full SQP approach reads as

$$\begin{bmatrix} \mathcal{L}_{yy} & \mathcal{L}_{yp} & h_x^\top & c_x^\top \\ \mathcal{L}_{py} & \mathcal{L}_{pp} & h_p^\top & c_p^\top \\ h_x & h_p & 0 & 0 \\ c_x & c_p & 0 & 0 \end{bmatrix} \begin{pmatrix} \Delta y \\ \Delta p \\ \Delta\mu \\ \Delta\lambda \end{pmatrix} = \begin{pmatrix} -\mathcal{L}_y^\top \\ -\mathcal{L}_p^\top \\ -h \\ -c \end{pmatrix}$$
$$\begin{pmatrix} y^{k+1} \\ p^{k+1} \\ \mu^{k+1} \\ \lambda^{k+1} \end{pmatrix} = \begin{pmatrix} y^k \\ p^k \\ \mu^k \\ \lambda^k \end{pmatrix} + \tau \cdot \begin{pmatrix} \Delta y \\ \Delta p \\ \Delta\mu \\ \Delta\lambda \end{pmatrix} \tag{23}$$

The symbol $\mathcal{L}$ denotes the Lagrangian function. We assume that the lift constraint h is active at the solution, which is the reason that we formulate it rather as an equality condition in the single set-point case. The approach in Eq. (23) is not implementable in general, because one usually starts out with a flow solver for $c(y, p) = 0$ and seeks a modular coupling with an optimization approach, which does not necessitate changing the whole code structure, as would be the case with formulation in Eq. (23). A modular, but nevertheless efficient, alternative is an approximate reduced SQP approach as justified in [17]

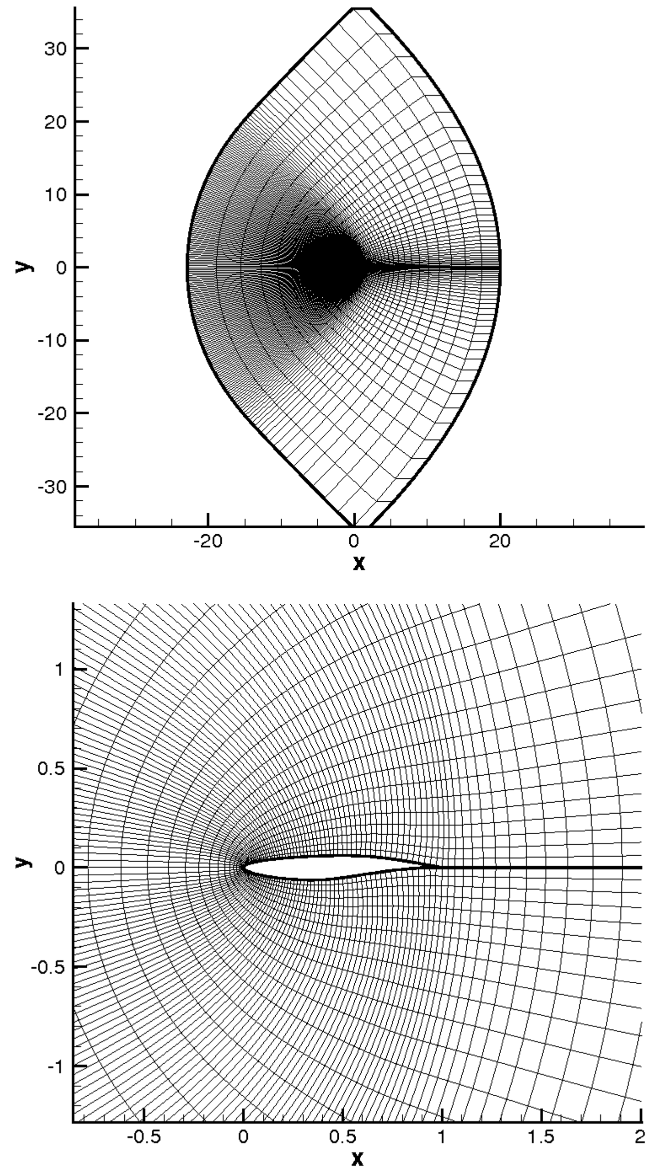

Fig. 2 Grid for the RAE2822 airfoil, showing the total geometrical plane (above) and zoom around the airfoil (below).

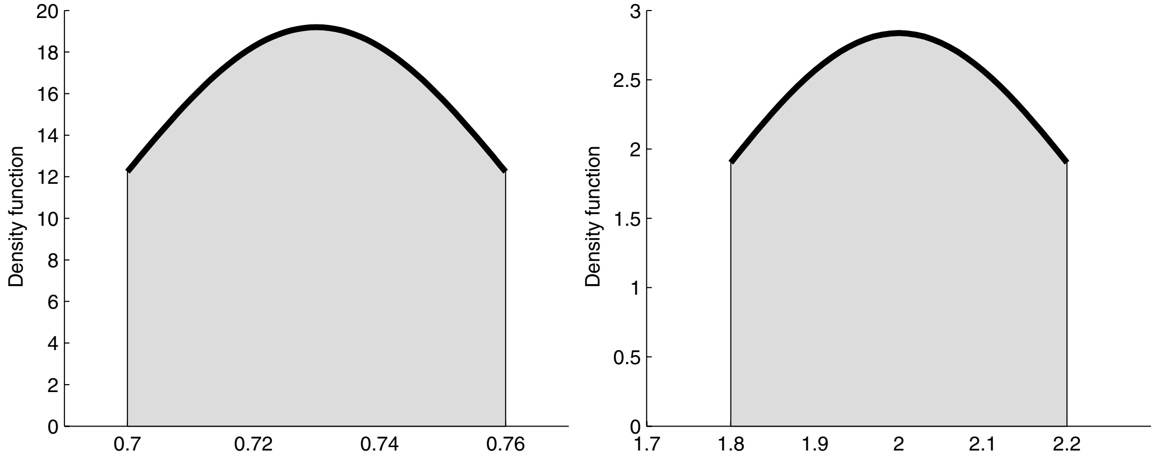

Fig. 3 Density function of the random variables $Ma \sim \frac{1}{\mathrm{const}}N(0.73, 0.001)\cdot \mathbf{1}_{[0.7,0.76]}$, $\alpha \sim \frac{1}{\mathrm{const}}N(2.0, 0.1)\cdot \mathbf{1}_{[1.8,2.2]}$.

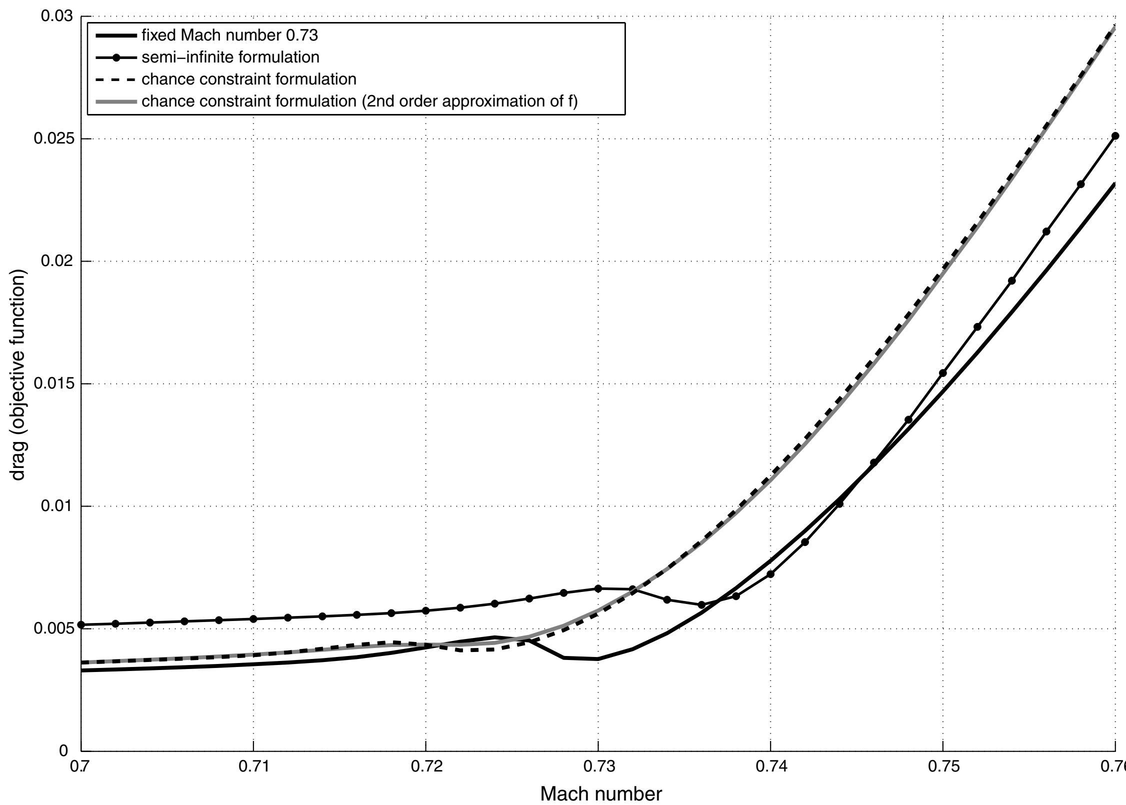

Fig. 4 Comparison of drag.

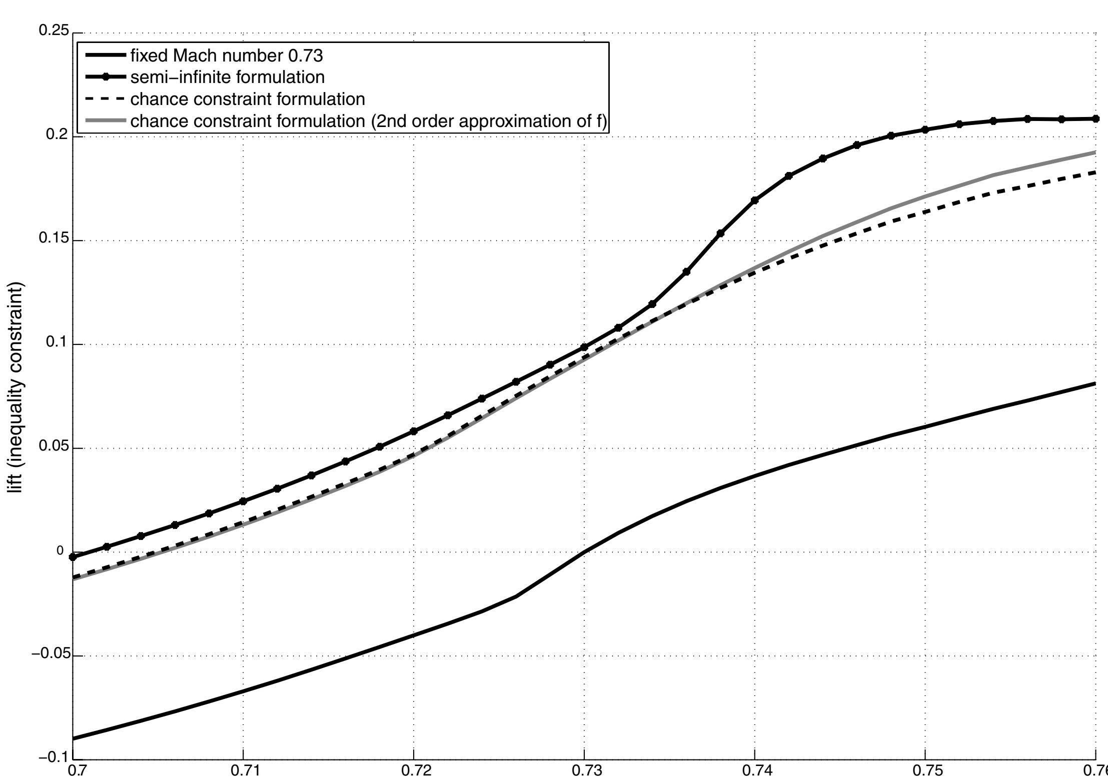

Fig. 5 Comparison of lift constraint.

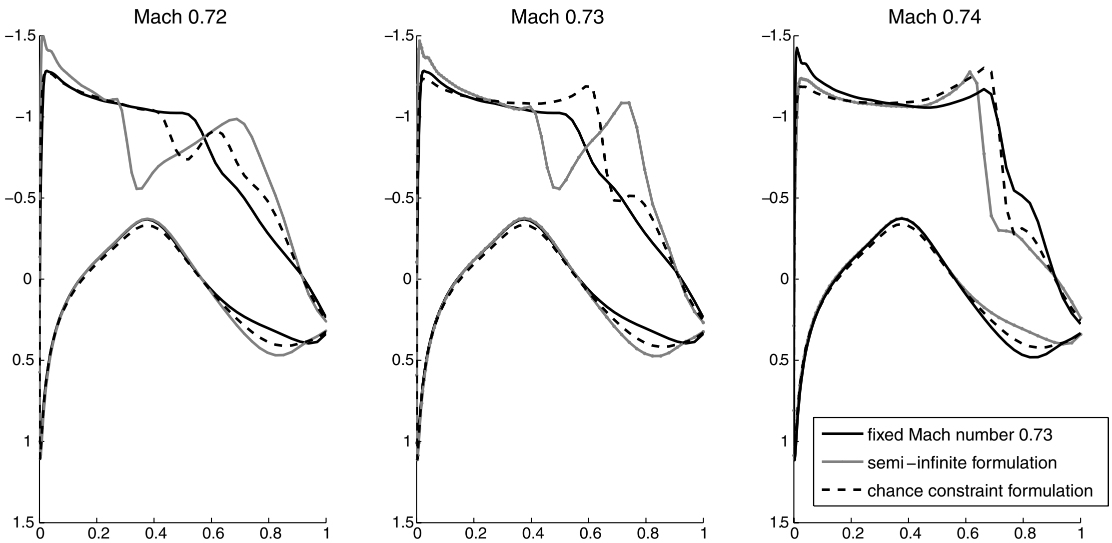

Fig. 6 Comparison of the distribution of the pressure coefficient over the airfoil.

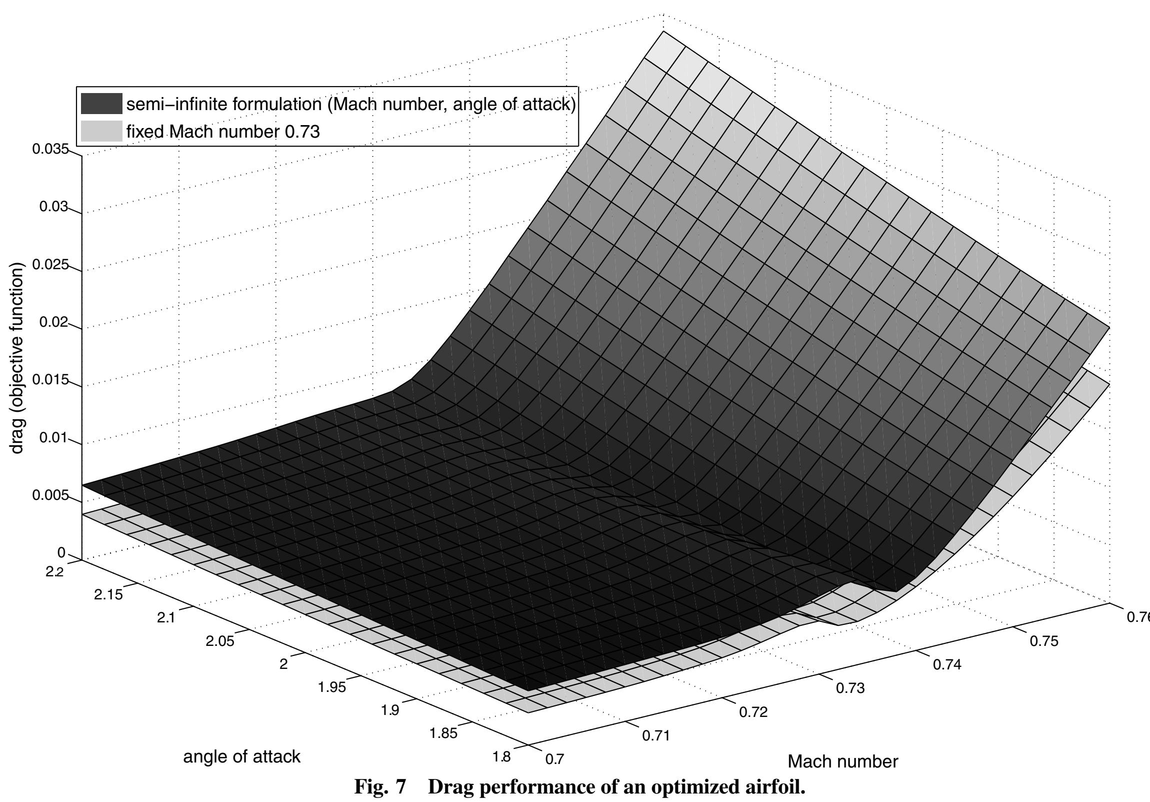

Fig. 7 Drag performance of an optimized airfoil.

$$\begin{bmatrix} 0 & 0 & 0 & A^\top \\ 0 & B & \gamma & c_p^\top \\ 0 & \gamma^\top & 0 & 0 \\ A & c_p & 0 & 0 \end{bmatrix} \begin{pmatrix} \Delta y \\ \Delta p \\ \Delta \mu \\ \Delta \lambda \end{pmatrix} = \begin{pmatrix} -\mathcal{L}_y^\top \\ -\mathcal{L}_p^\top \\ -h \\ -c \end{pmatrix}$$

$$\begin{pmatrix} y^{k+1} \\ p^{k+1} \\ \mu^{k+1} \\ \lambda^{k+1} \end{pmatrix} = \begin{pmatrix} y^k \\ p^k \\ \mu^k \\ \lambda^k \end{pmatrix} + \tau \cdot \begin{pmatrix} \Delta y \\ \Delta p \\ \Delta \mu \\ \Delta \lambda \end{pmatrix} \tag{24}$$

where

$$\gamma = h_p^\top + c_p^\top \alpha, \qquad \text{such that } A^\top \alpha = -h_x^\top$$

The matrix A denotes an appropriate approximation of the system matrix c_x, which is used in the iterative forward solver. An algorithmic version of this modular formulation is given by the following steps:

1) Generate λ^k by performing N iterations of an adjoint solver with right-hand-side $f_y^\top(y^k, p^k)$ starting in λ^k.

2) Generate α^k by performing N iterations of an adjoint solver with right-hand-side $h_y^\top(y^k, p^k)$ starting in α^k.

3) Compute approximate reduced gradients:

$$g = f_p^\top + c_p^\top \lambda^{k+1}, \qquad \gamma = h_p^\top + c_p^\top \alpha^{k+1}$$

4) Generate B_{k+1} as an approximation of the (consistent) reduced Hessian.

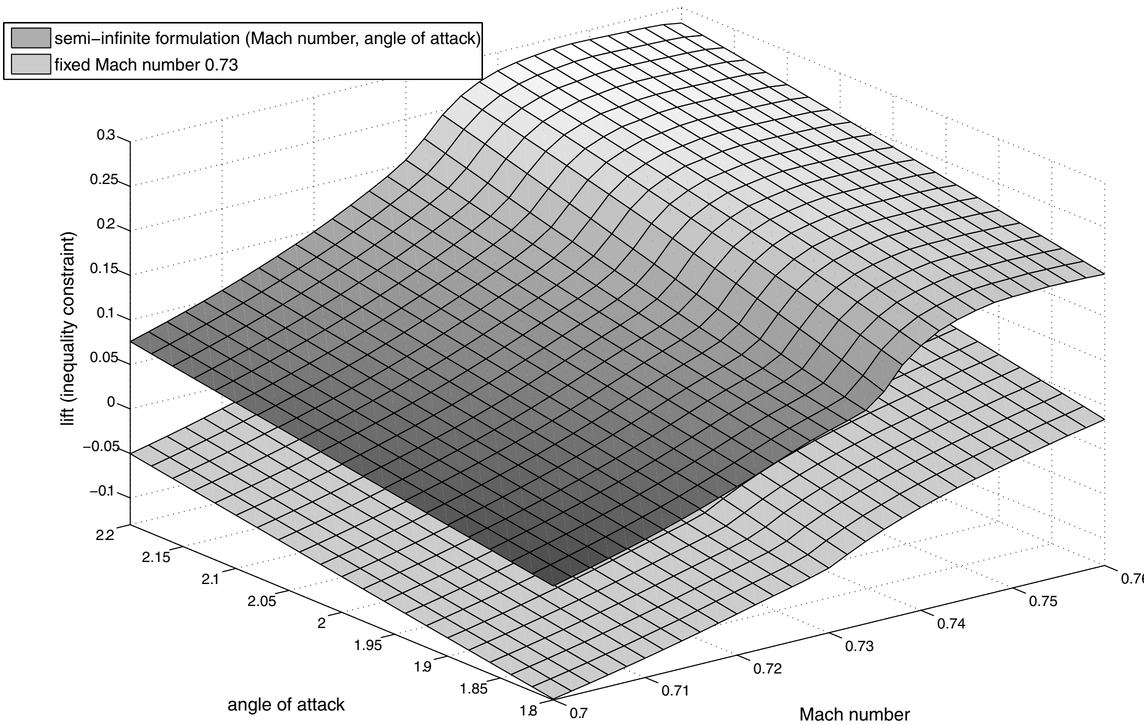

Fig. 8 Lift performance of an optimized airfoil.

5) Solve the quadratic program:

$$\begin{bmatrix} B & \gamma \\ \gamma^\top & 0 \end{bmatrix} \begin{pmatrix} \Delta p \\ \mu^{k+1} \end{pmatrix} = \begin{pmatrix} -g \\ -h \end{pmatrix}$$

6) Update $p^{k+1} = p^k + \Delta p$.

7) Compute the corresponding geometry and adjust the computational mesh.

8) Generate y^{k+1} by performing N iterations of the forward state solver starting from an interpolation of y^k at the new mesh. This highly modular algorithmic approach is not an exact transcription of Eq. (24), but is shown in [17] to be asymptotically equivalent and to converge to the same solution. The overall algorithmic effort for this algorithm is typically in the range of factor 7 to 10 compared with a forward stationary simulation.

Now we generalize this algorithmic framework to the semi-infinite problem formulation in Eqs. (10–12). Numerical approaches to this problem class have been proposed already in [18,19].

For the sake of simplicity, we restricted the formulation to a problem with two set points coupled via the objective, which is a weighted sum of all set-point objectives (weights: ω_1, ω_2), and via the free optimization variables p, which are the same for all set points. The generalization to more set points (i.e., four points in our numerical results) is then obvious. Furthermore, in the case of the restriction h being the lift, we know that it is monotonic in the Mach number. Therefore, it is enough to formulate the constraint for the smallest value $s_{\min}$. The corresponding Lagrangian in our example is

$$\mathcal{L}(y_1, y_2, p, \lambda_1, \lambda_2) = \sum_{i=1}^{2} \omega_i f_i(y_i, p, s_i) + \sum_{i=1}^{2} \lambda_i^\top c_i(y_i, p, s_i) + \mu h(y_{\min}, p, s_{\min}) \tag{25}$$

The preceding approximate reduced SQP method applied to this case can be written in the following form:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & A_1^\top & 0 \\ 0 & 0 & 0 & 0 & 0 & A_2^\top \\ 0 & 0 & B & \gamma_1 & c_{1,p}^\top & c_{2,p}^\top \\ 0 & 0 & \gamma_1 & 0 & 0 & 0 \\ A_1 & 0 & c_{1,p} & 0 & 0 & 0 \\ 0 & A_2 & c_{2,p} & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \Delta y_1 \\ \Delta y_2 \\ \Delta p \\ \Delta\mu \\ \Delta\lambda_1 \\ \Delta\lambda_2 \end{pmatrix} = \begin{pmatrix} -\mathcal{L}_{y_1}^\top \\ -\mathcal{L}_{y_2}^\top \\ -\mathcal{L}_p^\top \\ -h \\ -c_1 \\ -c_2 \end{pmatrix} \tag{26}$$

We notice that the linear subproblems involving matrices $A_i^\top$ are to be solved independently, and, therefore, trivially in parallel. The information from all these parallel adjoint problems is collected in the reduced gradient:

$$g = \sum_{i=1}^{2} \omega_i f_p^\top + \sum_{i=1}^{2} c_p^\top \lambda_i$$

Next, the solution of optimization step

$$\begin{bmatrix} B & \gamma_1 \\ \gamma_1^\top & 0 \end{bmatrix} \begin{pmatrix} \Delta p \\ \mu^{k+1} \end{pmatrix} = \begin{pmatrix} -g \\ -h \end{pmatrix}$$

is distributed to all approximate linearized forward problems:

$$A_i \Delta y_i + c_{i,p} \Delta p = -c_i$$

which can then again be performed in parallel.

V. Numerical Results

We investigate the problem discussed in [16], that is, the shape optimization of an RAE2822 profile in transonic Euler flow, by the use of the CFD software FLOWer provided by DLR within a one-shot framework. The block-structured FLOWer code solves the three-dimensional compressible Reynolds-averaged Navier-Stokes equation in integral form and provides different turbulence models. The equations are solved by a finite-volume method with second-order upwind or central space discretization. The discrete equations are integrated explicitly by multistage Runge–Kutta schemes, using local time stepping and multigrid acceleration. In our example, the

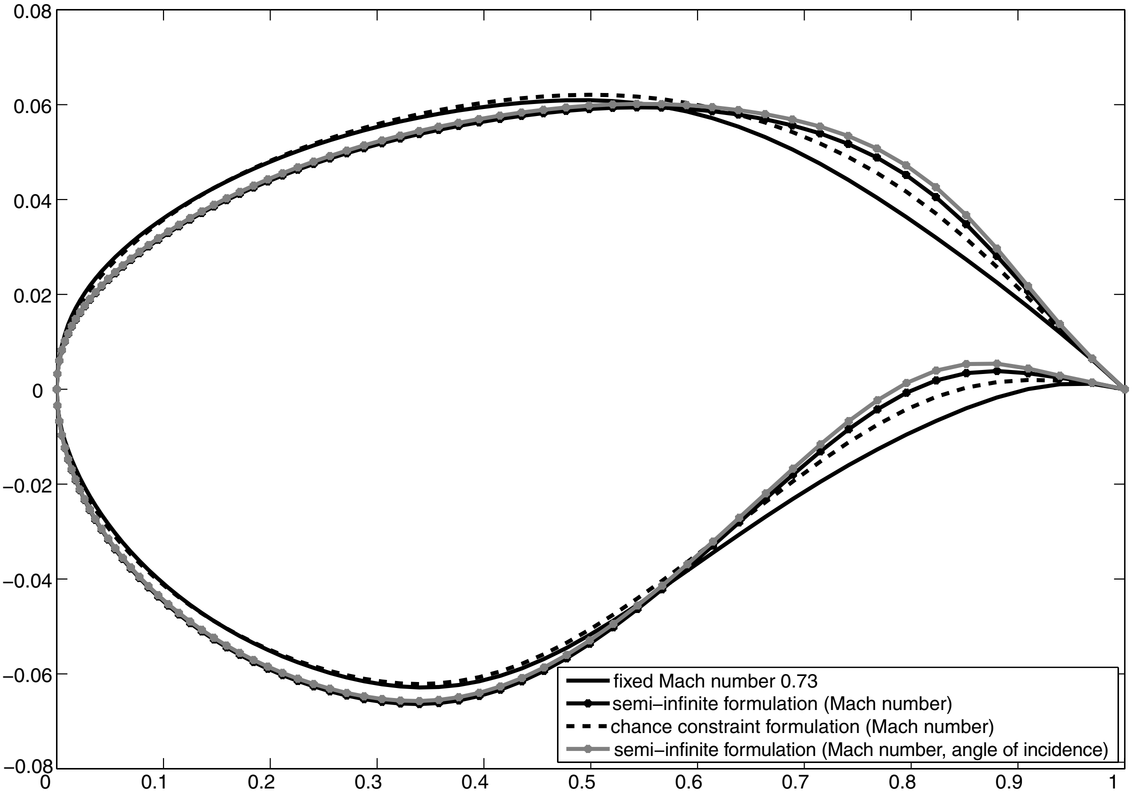

Fig. 9 Comparison of optimized shapes.

space is discretized by a 133×33 grid, see Fig. 2. For parameterization, the airfoil is decomposed into thickness and camber distribution. Then, only the camber of the airfoil is parameterized by 21 Hicks–Henne functions, and the thickness is not changed during the optimization process.

In this section, we consider the Mach number and the angle of attack as an uncertain parameter. The Mach number is assumed to be in the range of [0.7, 0.76] and the angle of attack in the range of [1.8, 2.2]. Under the assumption of truncated normally distributed parameters, we obtain the density functions shown next in Fig. 3.

At first, we perform numerical comparisons between a single set-point problem formulation at the set point $s^0 = 0.73$ Mach with the robust formulations in Secs. III.B and III.C. In particular, we compare four formulations: 1) nonrobust optimization at the Mach number 0.73 (fixed Mach number 0.73), 2) semi-infinite formulation of Eqs. (13–15), 3) chance constraint formulation of Eqs. (21) and (22) without higher-order terms in the objective, and 4) with higher-order terms in the objective.

Figure 4 shows the evaluations of the objective (drag) in these cases, and Fig. 5 shpes the constraint (lift).

We state the following observations: first, the higher-order terms in the objective of the chance constraint formulation in Eqs. (21) and (22) seem to make no difference because the two resulting drag curves are almost the same (cf. Fig. 4), which means that they can be safely omitted. Second, the semi-infinite robust formulation has a better lift-to-drag ratio than the chance constraint formulation, in particular in the region above the set point 0.73, due to the fact that the semi-infinite formulation shows a higher lift over the whole range of variations (cf. Fig. 5) and for the greater part a better drag performance than the chance constraint (cf. Fig. 4). The distribution of the pressure coefficient over the airfoil at three different Mach numbers is depicted in Fig. 6. At the set point 0.73, the distribution of the pressure coefficient of the semi-inifinite formulation shows a shock wave over the upper surface of the profile, which produces a higher drag than the other solutions. But we can observe that the semi-infinite formulation will lead to the best distribution if the Mach number deviates from the set point. So, the semi-infinite formulations lead to a robust solution, which gives a little bit higher drag at the nominal point but a better performance over the whole range of variations.

Furthermore, we consider the angle of attack as an additional uncertain parameter. Figure 7 shows the drag performance of the solution of the semi-infinite optimization problem compared with the solution of the single set-point optimization.

The semi-infinite formulation gives a higher drag over the whole range of variations than the usual single set-point case, but the solution of the semi-infinite formulation is always feasible, as required; whereas the single set-point optimization achieves the given lift only in a small region of the variations (cf. Fig. 8). In summary, the semi-infinite formulation leads, once again, to a better lift-to-drag ratio just as in the one-dimensional stochastic case.

The different optimized shapes are shown in Fig. 9. The semi-infinite formulation differs the most from the single set-point case due to the requirement of feasibility over the whole range of uncertainties.

VI. Conclusions

Robust design is an important task to make aerodynamic shape optimization relevant for practical use. It is also highly challenging because the resulting optimization tasks become much more complex than in the usual single set-point case. Essentially, two robust optimization formulations are compared in this paper. Because of the fact that the inequality constraint is almost linear in the uncertain parameter, the solution of the chance constraint and of the semi-infinite formulation are quite similar. Nevertheless, the discretized semi-infinite formulation seems to be of advantage in a numerical test case close to a real configuration.

Acknowledgment

This research has been supported by the German Federal Ministry of Economics and Labour within the collaborative effort of the Management and Minimization of Uncertainties in Numerical Aerodynamics or MUNA.

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